HARMONIC UNIVALENT FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE OPERATOR

Y. P. Yadav
Department of Mathematics
S. B. Patil College of Engineering, Indapur,
Indapur, Pune, (M.S) India 413106

V. B. Satpute
Department of Mathematics
S. B. Patil College of Engineering, Indapur,
Indapur, Pune, (M.S) India 413106

ABSTRACT.

A subclass of harmonic univalent functions is defined using generalized derivative operator and we have obtained among others results like, coefficient inequalities, distortion theorem and convex combination.

Keywords: Univalent functions, Harmonic functions, Derivative operator, Convex Combinations, Distortion Bounds

1. INTRODUCTION

A continuous function $f(z)$ is said to be a complex-valued harmonic function in a simply connected domain D in complex plane C if both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are real harmonic in D. Such functions can be expressed as

$$f(z) = h(z) + \overline{g(z)} \tag{1.1}$$

where $h(z)$ and $g(z)$ are analytic in D. We call $h(z)$ as analytic part and $g(z)$ as co-analytic part of $f(z)$. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ for all z in D. [2]

Let S_H be the family of functions of the form (1.1) that are harmonic, univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$, so that $f(z) = h(z) + \overline{g(z)}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Further $f(z) = h(z) + \overline{g(z)}$ can be uniquely determined by the coefficients of power series expansions.

$$h(z) = z + \sum_{p=2}^{\infty} a_p z^p, \qquad g(z) = \sum_{p=1}^{\infty} b_p z^p, \quad z \in U, \quad |b_1| < 1, \tag{1.2}$$

where $a_p \in C$ for $p = 2,3,4,...$ and $b_p \in C$ for $p = 1,2,3,...$

We note that this family S_H was investigated and studied by Clunie and Sheil-Small [2] and it reduces to the well-known family S the class of all normalized analytic univalent functions $h(z)$ given in (1.2), whenever the co-analytic part $g(z)$ of $f(z)$ is identically zero.

Let $\overline{S_H}$ denote the subfamily of S_H consisting of harmonic functions of the form

$$f_n(z) = h(z) + \overline{g_n(z)}$$

Where

$$\mathrm{h(z)=z+}\sum_{p=2}^{\infty} \mathrm{a_p} z^p, \quad \mathrm{g_n(z)=(-1)^n} \sum_{p=1}^{\infty} b_p z^p, \quad z \in U, \quad |b_1| < 1. \tag{1.3}$$

For $f(z) = h(z) + \overline{g(z)}$ given by(1.1), we define the derivative operator introduced by Shaqsi and Darus [8] of $f(z)$ as,

$$D^n_{m,\lambda} f(z) = D^n_{m,\lambda} h(z) + (-1)^n \overline{D^n_{m,\lambda} g(z)}, \tag{1.4}$$

where

$$D^n_{m,\lambda} h(z) = z + \sum_{p=2}^{\infty} \left[1 + (p-1)\lambda\right]^n C(m,p) a_p z^p$$

$$D_{m,\lambda}^{n}g(z)=\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}C(m,p)b_{p}z^{p},\ |b_{1}|<1,\ \ C(m,p)=C\binom{p+m-1}{m}.$$

Definition: The function $f(z)=h(z)+\overline{g(z)}$ defined by (1.2) is in the class $S_{H}(n,m,k,\lambda,\beta)$ if

$$\operatorname{Re}\left\{\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}\right\}\geq k\left|\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}-1\right|+\beta \tag{1.5}$$

where $0\leq k<\infty$, $0\leq\beta<1$.

Also let

$$\overline{S_{H}}(n,m,k,\lambda,\beta)=S_{H}(n,m,k,\lambda,\beta)\cap\overline{S_{H}} \tag{1.6}$$

We note that by specializing the parameter, especially when $k=0$, $S_{H}(n,m,k,\lambda,\beta)$ reduces to well-known family of starlike harmonic functions of order β . In recent years many researchers have studied various subclasses of S_{H} for example [1],[3],[4],[6]and [8].

In the present paper we aim at systematic study of basic properties, in particular coefficient bound , distortion theorem and extreme points of aforementioned subclass of harmonic functions.

2. MAIN RESULTS

Theorem1: Let $f(z)=h(z)+\overline{g(z)}$ be given by (1.2). If condition

$$\sum_{p=1}^{\infty}\left\{\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)|a_{p}|+\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)|b_{p}|\right\}\leq 2 \tag{2.1}$$

where $a_{1}=1,\ 0\leq\beta<1,\ \ 0\leq k<\infty,\ \ n\in N\cup\{0\}$,
then $f(z)$ is sense-preserving harmonic univalent in U and $f\in S_{H}(n,m,k,\lambda,\beta)$.

Proof: If the inequality (2.1) holds for coefficients of $f(z)=h(z)+\overline{g(z)}$ then by (1.2), $f(z)$ is orientation preserving and harmonic univalent in U. Now it remains to show that $f\in S_{H}(n,m,k,\lambda,\beta)$. According to (1.4) and (1.5) we have

$$\operatorname{Re}\left\{\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}\right\} \geq k\left|\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}-1\right|+\beta$$

which is equivalent to $\operatorname{Re}\left(\frac{A(z)}{B(z)}\right) > \beta$

where $A(z) = (1+k)D_{m,\lambda}^{n+1}f(z) - kD_{m,\lambda}^{n}f(z)$ and $B(z) = D_{m,\lambda}^{n}f(z)$

Using the fact that, $\operatorname{Re}(w) > \beta$ if $|1-\beta+w| \geq |1+\beta-w|$ it suffices to show that

$|A(z)+(1-\beta)B(z)| \geq |A(z)-(1+\beta)B(z)|$ substituting values of A(z) and B(z) with simple calculations we led to

$$= \left|(2-\beta)z + \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k+1-\beta\right]C(m,p)a_{p}z^{p} - (-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-1+\beta\right]C(m,p)\bar{b}_{p}\bar{z}^{p}\right|$$
$$-\left|\beta z + \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k+1-\beta\right]C(m,p)a_{p}z^{p} + (-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-1+\beta\right]C(m,p)\bar{b}_{p}\bar{z}^{p}\right|$$

$$\geq 2(1-\beta)|z| - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[2(1+k)\left[1+(p-1)\lambda\right]-2k-2\beta\right]C(m,p)|a_{p}||z|^{p}$$
$$-(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\left[2(1+k)\left[1+(p-1)\lambda\right]+2k+2\beta\right]C(m,p)|\bar{b}_{p}||\bar{z}|^{p}$$

$$\geq 2(1-\beta)|z|\left\{1 - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^{n}\frac{\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)|a_{p}||z|^{p-1}\right.$$
$$\left.-(-1)^{n}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^{n}\frac{\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)|\bar{b}_{p}||\bar{z}|^{p-1}\right\} \geq 0.$$

By assumption. Hence proof is completed.

The functions

$$f(z)=z+\sum_{p=2}^{\infty}\left[\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}\right]x_{p}z^{p}+\sum_{p=1}^{\infty}\left[\frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}\right]\overline{y}_{p}\overline{z}^{p}$$

where $\sum_{p=2}^{\infty}\left|x_{p}\right|+\sum_{p=1}^{\infty}\left|y_{p}\right|=1$ (2.3)

shows that the coefficient bound given (2.1) is sharp.

Theorem 2: Let $f_{n}(z)=h(z)+\overline{g_{n}(z)}$ be so that $h(z)$ and $g_{n}(z)$ given by (1.6). Then $f_{n}\in\overline{S_{H}}(n,m,k,\lambda,\beta)$ if and only if

$$\sum_{p=1}^{\infty}\left\{\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}C(m,p)\left|a_{p}\right|+\frac{\left[1+(p-1)\lambda\right]^{n}\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}C(m,p)\left|\overline{b}_{p}\right|\right\}\leq 2, \tag{2.4}$$

where $a_{1}=1, 0\leq\beta<1, 0\leq k<\infty.$

Proof: The if part follows form Theorem1 with the fact the $\overline{S_{H}}(n,m,k,\lambda,\beta)\subset S_{H}(n,m,k,\lambda,\beta)$. For only if part, we show that $f_{n}\notin\overline{S_{H}}(n,m,k,\lambda,\beta)$ if the condition (2.4) is not satisfied. Note that necessary and sufficient condition for Let $f_{n}=h+\overline{g_{n}}$ given by (1.6) to be in $\overline{S_{H}}(n,m,k,\lambda,\beta)$ is that

$$\operatorname{Re}\left\{\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}\right\}\geq k\left|\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}-1\right|+\beta$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{(1+k)D_{m,\lambda}^{n+1}f(z)+(k-\beta)D_{m,\lambda}^{n}f(z)}{D_{m,\lambda}^{n}f(z)}\right\}$$

$$= \operatorname{Re}\left\{ \frac{\begin{array}{l}(1-\beta)z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]C(m,p)a_p z^p \\ -(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]C(m,p)\bar{b}_p \bar{z}^p\end{array}}{\begin{array}{c}z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)a_p z^p \\ +(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)\bar{b}_p \bar{z}^p\end{array}} \right\} > 0.$$

The above conditions must hold for all values of z , $|z| = r < 1$. Choosing z on positive axis where $0 \le |z| = r < 1$. we have

$$\frac{\begin{array}{l}(1-\beta)z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]C(m,p)a_p r^{p-1} \\ -(-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]C(m,p)\bar{b}_p \overline{r^{p-1}}\end{array}}{z - \sum_{p=2}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)a_p r^{p-1} + (-1)^{2k}\sum_{p=1}^{\infty}\left[1+(p-1)\lambda\right]^n C(m,p)\bar{b}_p \overline{r^{p-1}}} \ge 0. \quad (2.5)$$

or equivalently if the condition (2.4) dose not hold then the numerator in (2.5) is negative for r sufficiently close to 1.

Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (2.5) is negative .This contradicts that required condition for $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ and hence proof is completed.

Theorem 3: Let f_n be given by (1.6). Then $f_n \in \overline{S_H}(k,\beta;n)$ if and only if

$$f_n(z) = \sum_{p=1}^{\infty}\left(x_p h_p(z) + y_p g_{n_p}(z)\right)$$

where, $h_1(z) = 1$,

$$h_p(z) = z - \frac{(1-\beta)}{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]} z^p, \quad p = 2,3,4,...$$

$$g_{n_p}(z) = z + (-1)^{n-1} \frac{(1-\beta)}{[1+(p-1)\lambda]^n [(1+k)[1+(p-1)\lambda]+k+\beta]} z^p, \qquad p = 1,2,3,... \text{ and}$$

$x_p \geq 0, y_p \geq 0$, $x_1 = 1 - \sum_{p=2}^{\infty}(x_p + y_p) \geq 0$.

In particular, the extreme points of $\overline{S_H}(n,m,k,\lambda,\beta)$ are $\{h_n\}$ and $\{g_{n_p}\}$.

Proof: Let

$$f_n(z) = \sum_{p=1}^{\infty}\left(x_p h_p(z) + y_p g_{n_p}(z)\right)$$

$$= \sum_{p=2}^{\infty}(x_p + y_p) - \sum_{p=2}^{\infty} \frac{(1-\beta)}{[1+(p-1)\lambda]^n [(1+k)[1+(p-1)\lambda]-k-\beta]} x_p z^p$$

$$+(-1)^{n-1} \sum_{p=1}^{\infty} \frac{(1-\beta)}{[1+(p-1)\lambda]^n [(1+k)[1+(p-1)\lambda]+k+\beta]} y_p \overline{z}^p$$

Then

$$= \sum_{p=2}^{\infty} \frac{[1+(p-1)\lambda]^n [(1+k)[1+(p-1)\lambda]-k-\beta]}{(1-\beta)} |a_p|$$

$$+ \sum_{p=1}^{\infty} \frac{[1+(p-1)\lambda]^n [(1+k)[1+(p-1)\lambda]+k+\beta]}{(1-\beta)} |b_p|$$

$$= \sum_{p=2}^{\infty} x_p + \sum_{p=1}^{\infty} y_p = 1 - x_1 \leq 1$$

and so $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$.

Conversely, suppose that $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$.

Setting

$$x_p = \frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}a_p, p=2,3,...$$

$$y_p = \frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}b_p, p=1,2,3,...$$

where $\sum_{p=1}^{\infty}\left(x_p+y_p\right)=1$ we obtain $f_n(z)=\sum_{p=1}^{\infty}\left(x_p h_p(z)+y_p g_{n_p}(z)\right)$ as required.

Theorem 4: Let $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ then for $|z|=r<1$

we have

$$|f_n(z)| \le (1+|b_1|)r + \frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta} - \frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\}r^2$$

and

$$|f_n(z)| \ge (1-|b_1|)r - \frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta} - \frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\}r^2$$

Proof. *Let* $f_n \in \overline{S_H}(n,m,k,\lambda,\beta)$ .Taking absolute value of f_n we obtain

$$|f_n(z)| \le (1+|b_1|)r + \sum_{p=2}^{\infty}\left(|a_p|+|b_p|\right) r^p$$

$$\le (1+|b_1|)r + \sum_{p=2}^{\infty}\left(|a_p|+|b_p|\right) r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\left\{\sum_{p=2}^{\infty}\frac{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}{(1-\beta)}\left(|a_p|+|b_p|\right)\right\}r^2$$

$$\le (1+|b_1|)r + \frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\sum_{p=2}^{\infty}\left(\frac{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}{(1-\beta)}|a_p| + \frac{(2\lambda)^n\left[(1+k)(1+\lambda)+k+\beta\right]}{(1-\beta)}|b_p|\right)r^2$$

$$\leq \left(1+|b_1|\right)r+\frac{(1-\beta)}{(2\lambda)^n\left[(1+k)(1+\lambda)-k-\beta\right]}\sum_{p=2}^{\infty}\left(1-\frac{(2\lambda)^n\left[(1+k)(1+\lambda)+k+\beta\right]}{(1-\beta)}|b_p|\right)r^2$$

$$\leq \left(1+|b_1|\right)r+\frac{1}{(2\lambda)^n}\left\{\frac{(1-\beta)}{(1+k)(1+\lambda)-k-\beta}-\frac{(1+k)(1+\lambda)+k+\beta}{(1+k)(1+\lambda)-k-\beta}|b_1|\right\}r^2.$$

The forthcoming result follows from left hand inequality in Theorem 2.4.

Theorem 5: The class of $\overline{S_H}(n,m,k,\lambda,\beta)$ is closed under convex combination.

Proof: For $i=1,2,3,\ldots$ suppose $f_{n_i}(z)\in\overline{S_H}(n,m,k,\lambda,\beta)$ where

$$f_{n_i}=z-\sum_{p=2}^{\infty}|a_{ip}|z^p+(-1)^n\sum_{p=1}^{\infty}|b_{ip}|\overline{z}^p$$

then by Theorem 2

$$\sum_{p=2}^{\infty}\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}|a_{ip}|$$
$$+\sum_{p=2}^{\infty}\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}|b_{ip}|\leq 1. \qquad (2.6)$$

For $\sum_{i=1}^{\infty}t_i=1,\ 0\leq t_i\leq 1,$ the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{n_i}(z)=z-\sum_{p=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{ip}|\right)z^p+(-1)^n\sum_{p=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{ip}|\right)\overline{z}^p$$

hence by (2.6)

$$\sum_{p=2}^{\infty}\left(\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i|a_{ip}|\right)$$
$$+\sum_{p=1}^{\infty}\left(\frac{\left[1+(p-1)\lambda\right]^n\left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)}\right)\left(\sum_{i=1}^{\infty}t_i|b_{ip}|\right)$$

$$= \sum_{i=i}^{\infty} t_i \left(\sum_{p=2}^{\infty} \frac{\left[1+(p-1)\lambda\right]^n \left[(1+k)\left[1+(p-1)\lambda\right]-k-\beta\right]}{(1-\beta)} \left|a_{ip}\right| + \sum_{p=1}^{\infty} \frac{\left[1+(p-1)\lambda\right]^n \left[(1+k)\left[1+(p-1)\lambda\right]+k+\beta\right]}{(1-\beta)} \left|a_{ip}\right| \right)$$

$$\leq \sum_{i=i}^{\infty} t_i \leq 1$$

and therefore $\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{S_H}(n,m,k,\lambda,\beta)$

This completes the proof.

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