Bayes Estimation of Topp-Leone Exponential Distribution using Symmetric Loss Functions for Identical Priors

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Abstract: ***The objective of this paper is to estimate the parameters of Topp-Leone Exponential distribution using the Bayesian approach. Exponential, Gamma, Log-Normal, and Weibull distributions were used as priors for estimating both parameters under identical distributions. Lindley approximation technique was adopted to find the Bayes estimates. In this paper symmetric loss functions were used to estimate the parameters of Topp-Leone Exponential distribution using different loss functions. Simulated data are used to compare the different models. It is observed that Gamma prior is the most preferred and QLF is the most preferred loss function for the shape parameter.***

Keywords: Topp-Leone Exponential distribution; Bayesian; Lindley's approximation; Symmetric Loss Function.

1. Introduction

A Bayesian estimation is a non-classical approach to statistical inference and its applications are widespread in the world. An alternative to the Beta distribution is the Topp–Leone distribution which is a bounded J-shaped distribution. Numerous authors have examined this distribution. The exponential distribution proposed by Epstein [4] plays a significant part in actual life scenarios. The Topp-Leone distribution was proposed by Topp and Leone [16], who also discussed the bounded Topp-Leone distribution for empirical data using a J-shaped histogram as a powered band tool. Nadarajah and Kotz [12] determined some J-shaped distribution moments. Kotz and Seier [9] compared kurtosis of the Topp-Leone distribution and left triangular distribution and Genc [6] studied the moments of order statistics of this distribution. Al-Shomrani et al [1] presented the family of distributions by Topp-Leone, its properties, and applications. Mohammed et al. [11] discussed the comparison of Topp-Leone, Topp-Leone, and Topp-Leone exponential expansion models for ovarian cancer patient data. Noman Rashed [13] discusses the properties and applications of the Topp-Leone compound Rayleigh. Fatoki Olayode[5] discusses the moment generation function, survival function, and ordinal statistics of the Topp-Leone Rayleigh distribution. Hind Jawad Kadhim Albderi [7] discussed the survival function of the Topp-Leone exponential distribution with the application. This paper adopts the Bayesian approach for Topp Leone Exponential distribution to estimate the parameters using priors such as Exponential, Gamma, Log-Normal, and Weibull. The posterior distribution for the unknown shape parameter $\eta \; and$ scale parameter δ assumed to follow identical priors for Topp-Leone Exponential distribution and is derived using the following four priors shown in Table 1:

Table:1 Identical priors

Priors	***Identical priors***	
	Shape Parameter $\boldsymbol{\eta}$	*Scale Parameter* $\boldsymbol{\delta}$
Exponential	*Exponential*	*Exponential*
Gamma	*Gamma*	*Gamma*
Log-Normal	*Log-Normal*	*Log-Normal*
Weibull	*Weibull*	*Weibull*

2.Maximum Likelihood Estimation

The p.d.f of Topp-Leone Exponential distribution [1] is given by

$$f(x;\eta,\delta) = 2\eta\delta\, e^{-2\delta x}\left(1 - e^{-2\delta x}\right)^{\eta-1}\ ; x,\eta,\delta > 0 \tag{1}$$

with η as shape parameter and δ the scale parameter.

Then the likelihood function:

$$L(x;\eta,\delta) = (2\eta\delta)^n\, e^{-2\delta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1 - e^{-2\delta x_i}\right)^{\eta-1} \tag{2}$$

Taking log of likelihood equation (2) and differentiating $w.r.to.\eta$ and δ gives

$$\frac{\partial LogL}{\partial\eta} = \frac{n}{\eta} + \sum_{i=1}^{n} log\left(1 - e^{-2\delta x_i}\right) = 0 \tag{3}$$

$$\frac{\partial LogL}{\partial\delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n} x_i + (\eta - 1)\sum_{i=1}^{n} \frac{2x_i e^{-2\delta x_i}}{\left(1 - e^{-2\delta x_i}\right)} = 0 \tag{4}$$

The maximum likelihood estimates (MLEs) of η and δ, say $\hat{\eta}$ and $\hat{\delta}$, respectively, are the solution of the equations (3) and (4). Unfortunately, analytic solutions for η and δ are nothing the closed form. To estimate the parameters η and δ Newton Raphson method is used.

3.Priors and Posterior Distributions

For Topp Leone-Exponential distribution Priors namely Exponential (E) prior, Gamma(G)Prior, Log-Normal (LN) prior and Weibull (W) prior, by assuming that shape parameter η and scale parameter δ both having identical priors namely E-E, G-G, LN-LN and W-W. The posterior distribution with identical priors for the shape and scale parameters are discussed as follows:

3.1. Posterior Distribution for Topp Leone-Exponential distribution using Identical Priors

3.1.1 Exponential Prior

The joint prior distribution using Exponential priors for both η and δ $i.e.,\eta\sim exp(a_1)$ and $\delta\sim exp(a_2)$.

$$p_1(\eta,\delta) = a_1 a_2 e^{-(a_1\eta + a_2\delta)}\ ; a_1, a_2 > 0 \tag{5}$$

The joint posterior distribution of η and δ isgiven by:

$$\pi_1(\eta,\delta|x) = \frac{1}{C_1}\, e^{-(a_1\eta + a_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1 - e^{-2\delta x_i}\right)^{\eta-1} \tag{6}$$

where $C_1 = \int_0^\infty \int_0^\infty e^{-(a_1\eta + a_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^{n} x_i} \prod_{i=1}^{n}\left(1 - e^{-2\delta x_i}\right)^{\eta-1} d\eta d\delta$

3.1.2. Gamma Prior

The joint prior distribution using Gamma priors for both η and δ $i.e.,\eta\sim G(a_3, b_1)$ and $\delta\sim G(a_4, b_2)$.

$$p_2(\eta,\delta)=\frac{b_1b_2}{\Gamma a_3\Gamma a_4}\eta^{a_3-1}\delta^{a_4-1}e^{-(b1\eta+b_2\delta)};\ \eta,\delta,a_3,a_4,b_1,b_2>0 \tag{7}$$

The joint posterior distribution of η and δ is given by:

$$\pi_2(\eta,\delta|x)=\frac{1}{C_2}\eta^{a_3-1}\delta^{a_4-1}e^{-(b1\eta+b_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1} \tag{8}$$

where $C_2=\int_0^\infty\int_0^\infty \eta^{a_3-1}\delta^{a_4-1}e^{-(b1\eta+b_2\delta)}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$

3.1.3. Log-Normal Prior

The joint prior distribution using Log-Normal prior for both η and δ $i.e.,\eta\sim LN(a_5,b_3)$ and $\delta\sim LN(a_6,b_4)$.

$$p_3(\eta,\delta)=\frac{1}{\eta b_3\sqrt{2\pi_1}}e^{-\frac{(log\eta-a_5)^2}{2b_3^2}}\frac{1}{\delta b_4\sqrt{2\pi_2}}e^{-\frac{(log\delta-a_6)^2}{2b_4^2}},a_5,a_6,b_3,b_4>0 \tag{9}$$

The joint posterior distribution of η and δ is given by:

$$\pi_3(\eta,\delta|x)=\frac{1}{C_3}\frac{1}{\eta\delta}e^{-\frac{(log\eta-a_5)^2}{2b_3^2}}e^{-\frac{(log\delta-a_6)^2}{2b_4^2}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta \tag{10}$$

where $C_3=\int_0^\infty\int_0^\infty\frac{1}{\eta\delta}e^{-\frac{(log\eta-a_5)^2}{2b_3^2}}e^{-\frac{(log\delta-\ \ _6)^2}{2b_4^2}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$

3.1.4. Weibull Prior

The joint prior distribution using given by of η and δ $i.e.,\eta\sim W(a_7,b_5)$ and $\delta\sim W(a_8,b_6)$.

$$p_4(\eta,\delta)=\frac{a_7}{b_5^{a_7}}\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\frac{a_8}{b_6^{a_8}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}} \tag{11}$$

The joint posterior distribution of η and δ is given by:

$$\pi_4(\eta,\delta|x)=\frac{1}{C_4}\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta \tag{12}$$

where $C_4=\int_0^\infty\int_0^\infty\eta^{a_7-1}e^{-\left(\frac{\eta}{b_5}\right)^{a_7}}\delta^{a_8-1}e^{-\left(\frac{\delta}{b_6}\right)^{a_8}}(2\eta\delta)^n e^{-2\delta\sum_{i=1}^n x_i}\prod_{i=1}^n\left(1-e^{-2\delta x_i}\right)^{\eta-1}d\eta d\delta$

4.Bayes Estimates under Different Loss Functions

To estimate the parameters of Topp Leone Exponential distribution loss functions under Squared Error Loss Function (SELF) and Quadratic Loss Function (QLF) under symmetric loss function given in Table -2 are considered.

TABLE: 2 Bayes estimators under different loss functions

Loss Function Expression	***Bayes Estimator***	
	Parameter η	Parameter δ
$SELF=L((\hat{\eta}-\eta)=(\hat{\eta}-\eta)^2$	$\hat{\eta}_{SELF}=E(\eta\|x)$	$\hat{\delta}_{SELF}=E(\delta\|x)$
$QLF=L(\hat{\eta}-\eta)=\left(\frac{\hat{\eta}-\eta}{\eta}\right)^2$	$\hat{\eta}_{QLF}=\left(\frac{E(\eta^{-1}\|x)}{E(\eta^{-2}\|x)}\right)$	$\hat{\delta}_{QLF}=\left(\frac{E(\delta^{-1}\|x)}{E(\delta^{-2}\|x)}\right)$

To estimate the parameter of the joint posterior distribution of η and δ in the equations (6), (8), (10) and (12) cannot be solved analytically. Hence, the Lindley approximation method is considered [3]. The posterior expectation can be expressed as

$$E[u(\eta,\delta)|x]=\frac{\int u(\eta,\delta)\exp[L(\eta,\delta)+\rho(\eta,\delta)]\,d(\eta,\delta)}{\int\exp[L(\eta,\delta)+\rho(\eta,\delta)]\,d(\eta,\delta)} \tag{13}$$

where $u(\eta,\delta)$ is afunction of η and δ only,$L(\eta,\delta)$ is the log-likelihood and $\rho(\eta,\delta)$ is the log of joint prior of η and δ.

According to Lindley [10], if the sample size n is sufficiently large, the above equation can be approximately evaluated through:

$$I(x) = u(\hat{\eta},\hat{\delta}) + \frac{1}{2}\left[(u_{\eta\eta} + 2u_{\eta}\rho_{\eta})\sigma_{\eta\eta} + (u_{\delta\eta} + 2u_{\delta}\rho_{\eta})\sigma_{\delta\eta} + (u_{\eta\delta} + 2u_{\eta}\rho_{\delta})\sigma_{\eta\delta}\right.$$
$$+(u_{\delta\delta} + 2u_{\delta}\rho_{\delta})\sigma_{\delta\delta} + \frac{1}{2}\left[(u_{\eta}\sigma_{\eta\eta} + u_{\delta}\sigma_{\eta\delta})(L_{\eta\eta\eta}\sigma_{\eta\eta} + L_{\eta\delta\eta}\sigma_{\eta\delta} + L_{\delta\eta\eta}\sigma_{\delta\eta} + L_{\delta\delta\eta}\sigma_{\delta\delta})\right]$$
$$+\frac{1}{2}\left[(u_{\eta}\sigma_{\delta\eta} + u_{\delta}\sigma_{\delta\delta})(L_{\eta\eta\delta}\sigma_{\eta\eta} + L_{\eta\delta\delta}\sigma_{\eta\delta} + L_{\delta\eta\delta}\sigma_{\delta\eta} + L_{\delta\delta\delta}\sigma_{\delta\delta})\right] \quad (14)$$

$u_{\eta} = \frac{\partial u(\eta,\delta)}{\partial \eta}$; $u_{\delta} = \frac{\partial u(\eta,\delta)}{\partial \delta}$; $u_{\eta\eta} = \frac{\partial^2 u(\eta,\delta)}{\partial \eta^2}$; $u_{\delta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial \delta^2}$; $u_{\eta\delta} = \frac{\partial^2 u(\eta,\delta)}{\partial \eta \partial \delta}$;$\frac{\partial^2 LogL}{\partial \eta^2} = L_{\eta\eta}$and so on.

with the above defined expressions, the values of the estimates for Topp Leone Exponential distribution are as follows.

$$E\left[u(\hat{\eta},\hat{\delta})|x\right] = u(\hat{\eta},\hat{\delta}) + \frac{1}{2}\left[(u_{\eta\eta} + 2u_{\eta}\rho_{\eta})\sigma_{\eta\eta} + (u_{\delta\eta} + 2u_{\delta}\rho_{\eta})\sigma_{\delta\eta} + (u_{\eta\delta} + 2u_{\eta}\rho_{\delta})\sigma_{\eta\delta}\right.$$
$$+(u_{\delta\delta} + 2u_{\delta}\rho_{\delta})\sigma_{\delta\delta} + \frac{1}{2}\left[(u_{\eta}\sigma_{\eta\eta} + u_{\delta}\sigma_{\eta\delta})(S_1)\right] + \frac{1}{2}\left[(u_{\eta}\sigma_{\delta\eta} + u_{\delta}\sigma_{\delta\delta})(S_2)\right] \quad (15)$$

Where $S_1 = L_{\eta\eta\eta}\sigma_{\eta\eta} + L_{\delta\delta\eta}\sigma_{\delta\delta}S_2 = L_{\eta\delta\delta}\sigma_{\eta\delta} + L_{\delta\eta\delta}\sigma_{\delta\eta} + L_{\delta\delta\delta}\sigma_{\delta\delta}$

(i)The log joint prior density of Exponential prior is:

$$\rho(\eta,\delta) = log a_1 + log a_2 - a_1\eta - a_2\delta$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = -a_1 \quad (16)$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = -a_2 \quad (17)$$

(ii)The log joint prior density of Gamma prior is:

$$\rho(\eta,\delta) = (a_3 - 1)log\eta - b_1\eta + (a_4 - 1)log\delta - b_2\delta$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = \frac{a_3 - 1}{\eta} - b_1 \quad (18)$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = \frac{a_4 - 1}{\delta} - b_2 \quad (19)$$

(iii) The log joint prior density of Log-Normal prior is:

$$\rho(\eta,\delta) = \log\left(\frac{1}{\eta\delta}\right) - \frac{(log\eta - a_5)^2}{2b_3^2} - \frac{(log\delta - a_6)^2}{2b_4^2}$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = -\frac{1}{\eta} - \frac{log\eta - a_5}{\eta b_3^2} \quad (20)$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = -\frac{1}{\delta} - \frac{log\delta - a_6}{\delta b_4^2} \quad (21)$$

(iv) The log joint prior density of Weibull prior is:

$$\rho(\eta,\delta) = (a_7 - 1)log\eta + (a_8 - 1)log\delta - \left(\frac{\eta}{b_5}\right)^{a_7} - \left(\frac{\delta}{b_6}\right)^{a_8}$$

$$\rho(\eta) = \frac{\partial \rho(\eta,\delta)}{\partial \eta} = \frac{a_7 - 1}{\eta} - \frac{a_7}{b_5}\left(\frac{\eta}{b_5}\right)^{a_7 - 1} \quad (22)$$

$$\rho(\delta) = \frac{\partial \rho(\eta,\delta)}{\partial \delta} = \frac{a_8 - 1}{\delta} - \frac{a_8}{b_6}\left(\frac{\delta}{b_6}\right)^{a_8-1} \quad (23)$$

Lindley's Approximation of η and δ using SELF:

The Bayes estimate using SELF for the parameter η using Exponential, Gamma, Log-Normal, and Weibull priors using equation (14) is given by:

$$\hat{\eta}_{exp} = \hat{\eta} - (a_1)\sigma_{\eta\eta} - (a_2)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2 \quad (24)$$

$$\hat{\eta}_G = \hat{\eta} + \left(\frac{a_3 - 1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} + \left(\frac{a_4 - 1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2 \quad (25)$$

$$\hat{\eta}_{LN} = \hat{\eta} + \left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\eta\eta} + \left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2 \quad (26)$$

$$\hat{\eta}_W = \hat{\eta} + \left(\frac{a_7 - 1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\eta\eta} + \left(\frac{a_8 - 1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} + \frac{1}{2}\sigma_{\eta\eta}S_1 + \frac{1}{2}\sigma_{\delta\eta}S_2 \quad (27)$$

The Bayes estimate using SELF for the parameter δ using Exponential, Gamma, Log-Normal, and Weibull priors using equation (14) is given by:

$$\hat{\delta}_{exp} = \hat{\delta} - (a_1)\sigma_{\delta\eta} - (a_2)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (28)$$

$$\hat{\delta}_G = \hat{\delta} + \left(\frac{a_3 - 1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + \left(\frac{a_4 - 1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (29)$$

$$\hat{\delta}_{LN} = \hat{\delta} + \left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} + \left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (30)$$

$$\hat{\delta}_W = \hat{\delta} + \left(\frac{a_7 - 1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + \left(\frac{a_8 - 1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} + \frac{1}{2}\sigma_{\eta\delta}S_1 + \frac{1}{2}\sigma_{\delta\delta}S_2 \quad (31)$$

Lindley's Approximation of η and δ using QLF:

The Bayes estimate using QLF for the parameter η using Exponential, Gamma, Log-Normal, and Weibull priors using equation (14) is given by:

$$\hat{\eta}_{exp} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} + \frac{1}{\hat{\eta}^2}(a_1)\sigma_{\eta\eta} + \frac{1}{\hat{\eta}^2}(a_2)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} + \frac{2}{\hat{\eta}^3}(a_1)\sigma_{\eta\eta} + \frac{2}{\hat{\eta}^3}(a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (32)$$

$$\hat{\eta}_G = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (33)$$

$$\hat{\eta}_{LN} = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta}-a_5}{\hat{\eta}b_3^2}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta}-a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(-\frac{1}{\hat{\eta}} - \frac{log\hat{\eta}-a_5}{\hat{\eta}b_3^2}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(-\frac{1}{\hat{\delta}} - \frac{log\hat{\delta}-a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (34)$$

$$\hat{\eta}_W = \frac{\frac{1}{\hat{\eta}} + \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\eta\eta} - \frac{1}{\hat{\eta}^2}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} - \frac{1}{2\hat{\eta}^2}\sigma_{\eta\eta}S_1 - \frac{1}{2\hat{\eta}^2}\sigma_{\delta\eta}S_2}{\frac{1}{\hat{\eta}^2} + \frac{3}{\hat{\eta}^4}\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\eta\eta} - \frac{2}{\hat{\eta}^3}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} - \frac{1}{\hat{\eta}^3}\sigma_{\eta\eta}S_1 - \frac{1}{\hat{\eta}^3}\sigma_{\delta\eta}S_2} \quad (35)$$

The Bayes estimate using QLF for the parameter δ using Exponential, Gamma, Log-Normal, and Weibull priors using equation (14) is given by:

$$\hat{\delta}_{exp} = \frac{\frac{1}{\hat{\delta}} + \frac{1}{\hat{\delta}^2}(a_1)\sigma_{\delta\eta} + \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta} + \frac{1}{\hat{\eta}^2}(a_2)\sigma_{\delta\delta} - \frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1 - \frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2} + \frac{3}{\hat{\delta}^4}\sigma_{\delta\delta} + \frac{2}{\hat{\delta}^3}(a_1)\sigma_{\delta\eta} + \frac{2}{\hat{\delta}^3}(a_2)\sigma_{\eta\delta} - \frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1 - \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \tag{36}$$

$$\hat{\delta}_G = \frac{\frac{1}{\hat{\delta}} - \frac{1}{\hat{\delta}^2}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} + \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta} - \frac{1}{\hat{\delta}^2}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\delta\delta} - \frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1 - \frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2} + \frac{3}{\hat{\delta}^4}\sigma_{\delta\delta} - \frac{2}{\hat{\delta}^3}\left(\frac{a_3-1}{\hat{\eta}} - b_1\right)\sigma_{\delta\eta} - \frac{2}{\hat{\delta}^3}\left(\frac{a_4-1}{\hat{\delta}} - b_2\right)\sigma_{\eta\delta} - \frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1 - \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \tag{37}$$

$$\hat{\delta}_{LN} = \frac{\frac{1}{\hat{\delta}} - \frac{1}{\hat{\delta}^2}\left(-\frac{1}{\hat{\eta}} - \frac{\log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} + \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta} - \frac{1}{\hat{\eta}^2}\left(-\frac{1}{\hat{\delta}} - \frac{\log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\delta\delta} - \frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1 - \frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2} + \frac{3}{\hat{\delta}^4}\sigma_{\delta\delta} - \frac{2}{\hat{\delta}^3}\left(-\frac{1}{\hat{\eta}} - \frac{\log\hat{\eta} - a_5}{\hat{\eta}b_3^2}\right)\sigma_{\delta\eta} - \frac{2}{\hat{\delta}^3}\left(-\frac{1}{\hat{\delta}} - \frac{\log\hat{\delta} - a_6}{\hat{\delta}b_4^2}\right)\sigma_{\eta\delta} - \frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1 - \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \tag{38}$$

$$\hat{\delta}_W = \frac{\frac{1}{\hat{\delta}} + \frac{1}{\hat{\delta}^2}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta} + \frac{1}{\hat{\eta}^2}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\delta\delta} - \frac{1}{2\hat{\delta}^2}\sigma_{\eta\delta}S_1 - \frac{1}{2\hat{\delta}^2}\sigma_{\delta\delta}S_2}{\frac{1}{\hat{\delta}^2} + \frac{3}{\hat{\delta}^4}\sigma_{\delta\delta} + \frac{2}{\hat{\delta}^3}\left(\frac{a_7-1}{\hat{\eta}} - \frac{a_7}{b_5}\left(\frac{\hat{\eta}}{b_5}\right)^{a_7-1}\right)\sigma_{\delta\eta} + \frac{2}{\hat{\delta}^3}\left(\frac{a_8-1}{\hat{\delta}} - \frac{a_8}{b_6}\left(\frac{\hat{\delta}}{b_6}\right)^{a_8-1}\right)\sigma_{\eta\delta} - \frac{1}{\hat{\delta}^3}\sigma_{\eta\delta}S_1 - \frac{1}{\hat{\delta}^3}\sigma_{\delta\delta}S_2} \tag{39}$$

5. Simulation Study

To compare the performance of Bayes estimates under different loss functions for Topp-Leone Exponential distribution simulation study was carried out. Data sets of sizes n=20,50 and 100 representing small, moderate and large samples with hyper parameters $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = m = 1, b_1 = b_2 = c = 0.5, b_3 = b_4 = 1, b_5 = b_6 = 2$ and N=5000 replications have been generated from Topp-Leone Exponential distribution. All the results were carried out using R programming.

The results of the simulation study to estimate the shape and scale parameters for different loss functions and with identical priors are present in Tables 3-4.

Table 3: Bayes estimates of estimators along with their MSE under different loss functions when $\eta = 1$ and $\delta = 1$under different priors.

SAMPLE SIZES	LOSS FUNCTIONS	PARAMETERS	PRIORS			
			EXPONENTIAL	GAMMA	LOG-NORMAL	WEIBULL
20	SELF	η	1.09291 (0.09641)	0.97288 (0.04347)	1.29156 (0.27105)	1.21296 (0.21106)
		δ	1.07410 (0.08527)	0.97676 (0.05292)	1.24309 (0.17627)	1.17146 (0.14370)
	QLF	η	0.96796 (0.08942)	0.92390 (0.08180)	1.08339 (0.12108)	1.02701 (0.11167)
		δ	0.95742 (0.08349)	0.91499 (0.07843)	1.06588 (0.10197)	1.0124 (0.09843)
50	SELF	η	1.04583 (0.04260)	1.01199 (0.03448)	1.10859 (0.06027)	1.07967 (0.05356)
		δ	1.03372 (0.03570)	1.00211 (0.03079)	1.09345 (0.04732)	1.06533 (0.04284)
	QLF	η	0.98165 (0.03529)	0.95790 (0.03339)	1.03424 (0.04105)	1.00837 (0.03931)
		δ	0.97345 (0.03341)	0.950787 (0.03270)	1.02384 (0.03608)	0.99872 (0.03568)
100	SELF	η	1.02146 (0.01905)	1.00617 (0.01739)	1.05099 (0.02271)	1.03676 (0.02123)
		δ	1.01657 (0.01776)	1.00171 (0.01661)	1.04551 (0.02046)	1.03144 (0.01939)
	QLF	η	0.98822 (0.01722)	0.97531 (0.01677)	1.01541 (0.01853)	1.00191 (0.01813)
		δ	0.98436 (0.01711)	0.97179 (0.01696)	1.01096 (0.01774)	0.99766 (0.01766)

Table 4: Bayes estimates of estimators along with their MSE under different loss functions when $\eta = 1.5$ and $\delta = 1$under different priors.

SAMPLE SIZES	LOSS FUNCTIONS	PARAMETERS	PRIORS			
			EXPONENTIAL	GAMMA	LOG-NORMAL	WEIBULL
20	SELF	η	1.50400 (0.24165)	1.20433 (0.22721)	1.93595 (1.00113)	1.80366 (0.71758)
		δ	1.01352 (0.16683)	0.90117 (0.21333)	1.18652 (0.14987)	1.12587 (0.15647)
	QLF	η	1.40289 (0.28932)	1.32924 (0.24520)	1.58490 (0.45838)	1.51138 (0.39283)
		δ	0.93911 (0.21704)	0.89308 (0.23940)	1.04577 (0.17775)	1.00102 (0.19689)
50	SELF	η	1.54442 (0.24328)	1.46638 (0.17918)	1.66380 (0.36235)	1.62246 (0.32355)
		δ	1.01456 (0.14536)	0.97873 (0.15981)	1.07290 (0.12547)	1.05039 (0.13380)
	QLF	η	1.44456 (0.18442)	1.39751 (0.15780)	1.53463 (0.24589)	1.50023 (0.22428)
		δ	0.96772 (0.16894)	0.94239 (0.18155)	1.01651 (0.14672)	0.99639 (0.15653)
100	SELF	η	1.52233 (0.18616)	1.48785 (0.16343)	1.57654 (0.22508)	1.55682 (0.21156)
		δ	1.00765 (0.13637)	0.99089 (0.14386)	1.03554 (0.12459)	1.02441 (0.12948)
	QLF	η	1.46700 (0.15494)	1.43991 (0.14051)	1.51456 (0.18263)	1.49646 (0.17265)
		δ	0.98187 (0.14920)	0.96776 (0.15624)	1.00742 (0.13695)	0.99689 (0.14219)

Summary

To estimate the shape η and scale δ parameters of Topp-Leone Exponential distribution with identical prior selections (i.e.,) E-E, G-G, LN- LN, and W-W priors, the comparison is done based on different loss functions using Bayes estimation.

In simulation study, for Topp-Leone Exponential distribution for different sample sizes with constant scale parameter $\delta = 1$, it is observed from Table 3-4 that

(i) The loss functions QLF and SELF for shape and scale parameter performs better with $\eta = 1$ and 1.5.

(ii) Based on different priors with shape parameter $\eta = 1$ and δ=1the Gamma prior performs better with loss functions being QLF for shape parameter and SELF for scale parameter.

(iii) In the case of $\eta = 1.5$ and δ=1, the shape parameter with loss function QLF for Gamma prior performs better but for the scale parameter with loss function SELF performs better with Log-Normal prior.

7.Conclusion

In the case of Topp Leone Exponential distribution with shape parameter η and scale parameter δ both assuming identical priors it is observed that shape parameter value increases. Also, it is seen that Gamma prior is the most preferred prior and QLF is the most preferred loss function for the shape parameter for stimulated data study. Further studied are needed to conform the result with increasing sample size.

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